Appendix: Mathematical Derivations

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Appendix

Affine Transformation to the Line Space

In this section we derive the affine transformation for the line space of coordinates. Let us consider lines represented in *Plücker* coordinates. The line coordinates before the application of the affine transformation is denoted as $\mathbf{l}^{(1)}\mathbb{R} = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)}\right)$ and after the application of the transformation as $\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{d}^{(2)}, \mathbf{m}^{(2)}\right)$. From the definition of *Pücker* coordinates [1], one has

$$\mathbf{d}^{(1)} = \mathbf{p}_2^{(1)} - \mathbf{p}_1^{(1)} \quad \text{and} \quad \mathbf{m}^{(1)} = -\mathbf{p}_2^{(1)} \times \mathbf{p}_1^{(1)} \tag{1}$$

for any two points $\mathbf{p}_1^{(1)}, \mathbf{p}_2^{(1)} \in \mathbb{R}^3$ (regular coordinates) that belong to the line, before the application of the affine transformation.

Using the affine parameters $\{\mathbf{B}, \mathbf{b}, b\}$, a point $\mathbf{p}_i^{(2)}$ after the application of the affine transformation is given by

$$\mathbf{p}_i^{(2)} = b^{-1}\left(\mathbf{B}\mathbf{p}_i^{(1)} + \mathbf{b}\right) \tag{2}$$

where $\mathbf{B} \in \mathbb{R}^{3\times 3}$ and $\mathbf{b} \in \mathbb{R}^3$. From the definition of *Pücker* coordinates – Equation (1), we derive the following equations

$$\mathbf{d}^{(2)} = b^{-1}\left(\mathbf{B}\mathbf{p}_2^{(1)} + \mathbf{b}\right) - b^{-1}\left(\mathbf{B}\mathbf{p}_1^{(1)} + \mathbf{b}\right) = b^{-1}\mathbf{B}\mathbf{d}^{(1)} \tag{3}$$

and

$$\begin{aligned} \mathbf{m}^{(2)} &= -b^{-1}\left(\mathbf{B}\mathbf{p}_2^{(1)} + \mathbf{b}\right) \times b^{-1}\left(\mathbf{B}\mathbf{p}_1^{(1)} + \mathbf{b}\right) \\ &= -b^{-2}\left(\mathbf{B}\mathbf{p}_2^{(1)} \times \mathbf{B}\mathbf{p}_1^{(1)} + \mathbf{b} \times \left(\mathbf{B}\left(\mathbf{p}_1^{(1)} - \mathbf{p}_2^{(1)}\right)\right)\right). \end{aligned} \tag{4}$$

From the properties of the cross product, one has $\left(\mathbf{B}\mathbf{p}_2^{(1)}\right) \times \left(\mathbf{B}\mathbf{p}_1^{(1)}\right) = \det(\mathbf{B})\,\mathbf{B}^{-T}\left(\mathbf{p}_2^{(1)} \times \mathbf{p}_1^{(1)}\right)$. Using this result and from Equation (1), one obtains

$$\mathbf{m}^{(2)} = -b^{-2}\left(-\det(\mathbf{B})\,\mathbf{B}^{-T}\mathbf{m}^{(1)} - \mathbf{b} \times \mathbf{B}\mathbf{d}^{(1)}\right). \tag{5}$$

Using the results derived in Equations (3) and (5), we formalize the following Proposition.

Proposition 1 *Considering a line $\mathbf{l}^{(1)}\mathbb{R} = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)}\right) \subset \mathbb{R}^6$, represented in Plücker coordinates, and an affine transformation $\{\mathbf{B}, \mathbf{b}, b\}$ that verifies Equation (2). The line coordinates $\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{d}^{(2)}, \mathbf{m}^{(2)}\right) \subset \mathbb{R}^6$ after the application of the affine transformation are given by*

$$\mathbf{l}^{(2)}\mathbb{R} = \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ b^{-1}\widehat{\mathbf{b}}\mathbf{B} & b^{-1}\mathit{det}(\mathbf{B})\,\mathbf{B}^{-T} \end{bmatrix}}_{\mathbf{E}} \mathbf{l}^{(1)}. \tag{6}$$

Note that $b \neq 0$ and matrix $\mathbf{B}$ is a non–singular matrix. As a result, matrix $\mathbf{E}$ is invertible and $\mathbf{l}^{(1)}$ can be estimated such that $\mathbf{l}^{(1)}\mathbb{R} = \mathbf{E}^{-1}\mathbf{l}^{(2)}$.

In the case of an Euclidean transformation, one has $\mathbf{B} = \mathbf{R} \in \mathcal{SO}(3)$, $\mathbf{b} = \mathbf{t} \in \mathbb{R}^3$ and $b = 1$. Thus, we write the following Result.

Results 1 *Let us consider a line $\mathbf{l}^{(1)}\mathbb{R} = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)}\right) \subset \mathbb{R}^6$, represented in Plücker coordinates, and a rigid transformation $\{\mathbf{R}, \mathbf{t}\}$ where $\mathbf{R} \in \mathcal{SO}(3)$ and $\mathbf{t} \in \mathbb{R}^3$. The line coordinates $\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{d}^{(2)}, \mathbf{m}^{(2)}\right) \subset \mathbb{R}^6$ after the application of the transformation are given by*

$$\mathbf{l}^{(2)}\mathbb{R} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \widehat{\mathbf{t}}\mathbf{R} & \mathbf{R} \end{bmatrix} \mathbf{l}^{(1)}. \tag{7}$$

We note that this particular case was derived in [2], [3].

Geometric Distance Between a 3D Line and a 3D Point

In this subsection, we derive a geometric distance between a line in the world $\mathbf{l} \in \mathbb{L}^3$ and a non-incident 3D point $\mathbf{p} \in \mathbb{P}^3$. Let us consider that the line is represented in *Plücker* coordinates $\mathbf{l}\mathbb{R} = (\mathbf{d}, \mathbf{m}) \subset \mathbb{R}^6$, where $\mathbf{d} \in \mathbb{R}^3$ and $\mathbf{m} \in \mathbb{R}^3$ represents the moment and direction of the line respectively.

The distance between a line and a point does not change after a rigid transformation. As a result, instead of estimating the distance in the world coordinate system, the distance can be computed in any other coordinate system.

Let us consider a rigid transformation defined by the translation $\mathbf{t} \in \mathbb{R}^3$ and rotation $\mathbf{R} \in \mathcal{SO}(3)$. From Result 1 – Section A, the line $\mathbf{l}^{(1)}\mathbb{R} = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)}\right)$ can be represented in the new coordinate system $\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{d}^{(2)}, \mathbf{m}^{(2)}\right)$, using

$$\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{R}\mathbf{d}^{(1)}, \mathbf{R}\mathbf{m}^{(1)} + \widehat{\mathbf{t}}\mathbf{R}\mathbf{d}^{(1)}\right). \tag{8}$$

Let us consider a generic line in the world coordinate system $\mathbf{l}^{(1)} \in \mathbb{L}^3$ and a non–incident point $\mathbf{p} \in \mathbb{P}^3$. The coordinates of the same line can be represented in a new coordinate system centered at $\mathbf{p}$, setting $\mathbf{t} = -\mathbf{p}$ and $\mathbf{R} = \mathbf{I}$, as

$$\mathbf{l}^{(2)}\mathbb{R} = \left(\mathbf{d}^{(2)}, \mathbf{m}^{(2)}\right) = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)} - \widehat{\mathbf{p}}\mathbf{d}^{(1)}\right). \tag{9}$$

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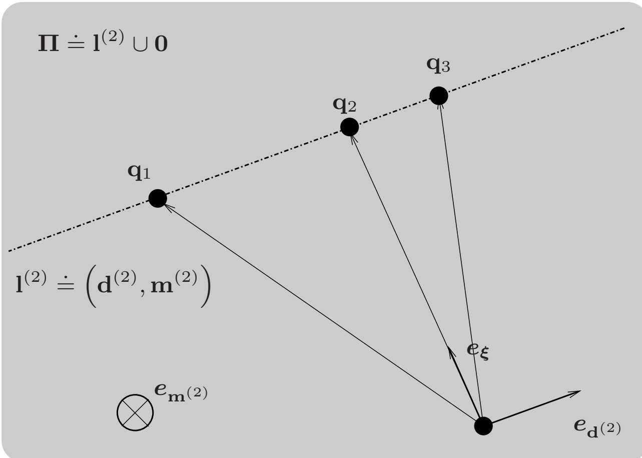

Fig. 1. In this figure we display the representation of a line $\mathbf{l}^{(1)}$ in the coordinate system centered at point $\mathbf{p}$. The plane $\boldsymbol{\pi}$ is defined by $\mathbf{l}^{(2)}$ and the origin of the coordinate system.

In this coordinate system, a plane $\mathbf{\Pi} \doteq \mathbf{l}^{(2)} \cup \mathbf{0}$, spanned by the line and the origin of the coordinate system, can be defined. In this coordinate system, any point $\mathbf{q}$ incident on the line $\mathbf{l}^{(2)}$ verifies

$$\mathbf{q} = \underbrace{\left(\mathbf{q} \cdot \boldsymbol{e}_{\mathbf{d}^{(2)}}\right)}_{\mathbf{q}_-} \boldsymbol{e}_{\mathbf{d}^{(2)}} + \underbrace{\left(\mathbf{q} \cdot \boldsymbol{e}_{\boldsymbol{\xi}}\right)}_{\mathbf{q}_+} \boldsymbol{e}_{\boldsymbol{\xi}} \tag{10}$$

where $\boldsymbol{e}_{\mathbf{d}^{(2)}}$ and $\boldsymbol{e}_{\boldsymbol{\xi}}$ are the orthogonal basis for the subspace $\mathbf{\Pi}$. The representation of this basis is show in Figure 1.

Since $\mathbf{q}_+$ is constant, vector $\mathbf{q}$ has the minimum norm when $\mathbf{q}_- = 0$ or $\mathbf{q} \cdot \boldsymbol{e}_{\mathbf{d}^{(2)}} = 0$ and, therefore, the distance between the line and the point can be defined as $\delta\left(\mathbf{l}, \mathbf{p}\right) = ||\mathbf{q}||$, such that $\mathbf{q} \cdot \boldsymbol{e}_{\mathbf{d}^{(2)}} = 0$.

From the definition of moment of the line, we have $\mathbf{m}^{(2)} = \mathbf{q} \times \mathbf{d}^{(2)}$ for any $\mathbf{q}$ that belong to the line, which means that

$$\mathbf{m}^{(2)} = ||\mathbf{q}|| \left|\left|\mathbf{d}^{(2)}\right|\right| \sin\left(\Theta\left(\mathbf{q}, \mathbf{d}^{(2)}\right)\right) \boldsymbol{e}_{\boldsymbol{\xi}} \times \boldsymbol{e}_{\mathbf{d}^{(2)}} \tag{11}$$

where $\Theta\left(\mathbf{q}, \mathbf{d}\right)$ is the angle between the vector $\mathbf{q}$ and $\mathbf{d}$. Note that the aim is to find $\mathbf{q}$ that verifies the constraint $\mathbf{q} \cdot \boldsymbol{e}_{\mathbf{d}^{(2)}} = 0$, which implies $\sin\left(\Theta\left(\mathbf{q}, \mathbf{d}\right)\right) = 1$ and, as a result,

$$\left|\left|\mathbf{m}^{(2)}\right|\right| = ||\mathbf{q}|| \left|\left|\mathbf{d}^{(2)}\right|\right|. \tag{12}$$

Without loss of generality, let us consider that $\mathbf{l}^{(1)} = \left(\mathbf{d}^{(1)}, \mathbf{m}^{(1)}\right)$ where $\left|\left|\mathbf{d}^{(1)}\right|\right| = 1$ which, from Equation (9), implies that $\left|\left|\mathbf{d}^{(2)}\right|\right| = 1$. The distance between a line and a point can be defined as

$$\delta\left(\mathbf{l}, \mathbf{p}\right) = ||\mathbf{q}|| = \left|\left|\mathbf{m}^{(2)}\right|\right|. \tag{13}$$

To conclude, using the Equations (13) and (9), we derive the following Proposition

Proposition 2 *For a line* $\mathbf{l}\mathbb{R} = \left(\mathbf{d}, \mathbf{m}\right) \subset \mathbb{R}^6$, *represented in Plücker coordinates, and a non-incident 3D point* $\mathbf{p} \in \mathbb{R}^3$, *the geometric distance between* $\mathbf{l}$ *and* $\mathbf{p}$ *are given by* $\delta\left(\mathbf{l}, \mathbf{p}\right)$, *such that*

$$\delta\left(\mathbf{l}, \mathbf{p}\right) = \frac{\left|\left|\left[\begin{array}{cc} -\widehat{\mathbf{p}} & \mathbf{I} \end{array}\right] \mathbf{l}^{(1)}\right|\right|}{\left|\left|\mathbf{d}^{(1)}\right|\right|}. \tag{14}$$

References

[1] H. Pottmann and J. Wallner, *Computational Line Geometry*. Berlin: Springer–Verlag, 2001.

[2] R. Pless, "Using Many Cameras as One," *In CVPR*, 2003.

[3] P. Sturm, "Multi–View Geometry for General Camera Models," *In CVPR*, 2005.