Generalized Essential Matrix: Properties of the Singular Value Decomposition

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Abstract

When considering non-central imaging devices, the computation of the relative pose requires the estimation of the rotation and translation that transform the 3D lines from one coordinate system to the second. In most of the state-of-the-art methods, this transformation is estimated by the computing a 6×6 matrix, known as the Generalized Essential Matrix. To allow a better understanding of this matrix, we derive some properties associated with its singular value decomposition.

Keywords: Generalized epipolar geometry, relative pose, singular value decomposition, rigid transformation of lines.

1. Introduction

Relative pose estimation is one of the main problems in computer vision, which has been studied for more than a century [1]. The goal is to estimate the rigid transformation between two cameras (or the same camera in two different positions) using matching between pixels that are images of the same 3D point in the world. The cameras (or camera) are considered calibrated. As a result, for each image pixel, we know the corresponding 3D projection line in the world. Thus, by computing the 3D projection lines associated to each match of pixels, the problem can be seen as finding the rotation and translation that align the 3D projection lines to ensure that they intersect in the world, as shown in Fig. 1. One of the most important applications is its use in robotics navigation, in methods such as visual odometry [2].

When considering conventional perspective cameras there are several solutions for the relative pose. We note that there are minimal (5-point algorithms) and non-minimal solutions. One of the goals of minimal solutions is to allow the determination of outliers from a large data-set, to build a robust data-set. On the other hand, the goal of non-minimal solutions is to estimate directly an accurate solution, from a given data-set. A common procedure is to run first the minimal solutions using RANSAC [3, 4], followed by iterative refinement, using non-minimal methods. In most of the approaches, authors used the essential matrix [5]. Let us consider a rotation matrix $\mathbf{R} \in \mathcal{SO}(3)$ and a translation vector $\mathbf{t} \in \mathbb{R}^3$, from the epipolar geometry one has

$$\mathbf{d}_i^{(2)\,T} \underbrace{s(\mathbf{t})\,\mathbf{R}}_{\mathbf{E}} \mathbf{d}_i^{(1)} = 0, \quad \text{where} \quad s(\mathbf{t}) \subset \mathbb{R}^{3\times 3} \doteq \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}, \tag{1}$$

where $\mathbf{d}_i^{(1)}$ and $\mathbf{d}_i^{(2)}$ denote the inverse projection of two pixels that are the images of the same 3D points with distinct cameras with different external parameteres–see Fig. 1(a). Matrix $\mathbf{E} \subset \mathbb{R}^{3\times 3}$ is known as the essential matrix. Some properties associated with the singular value decomposition of $\mathbf{E}$ were derived in [6, 7, 8]:

Proposition 1: *The essential matrix* $\mathbf{E}$ *is such that* $\mathbf{E}\mathbf{E}^T$ *only depends on the translation vector* $\mathbf{t}$ *and the singular value decomposition of* $\mathbf{E}\mathbf{E}^T$ *has one singular value equal to zero and other two singular values are equal.*

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Based on properties of its singular value decomposition, one can define the following constraints:

Proposition 2: $\mathbf{E}$ *is an essential matrix (which means that it can be decomposed into rotation and translation) if and only if*

$$det\left(\mathbf{E}\right) = 0 \quad and \quad \frac{1}{2} tr\left(\mathbf{E}\mathbf{E}^T\right)^2 - tr\left(\left(\mathbf{E}\mathbf{E}^T\right)^2\right) = 0.$$

In addition, the following constraint can also be derived

$$\frac{1}{2} tr\left(\mathbf{E}\mathbf{E}^T\right)\mathbf{E} - \mathbf{E}\mathbf{E}^T\mathbf{E} = 0.$$

These constraints (which ensure that $\mathbf{E}$ can be decomposed into rotation and translation in the way shown in (1)) were used in most of the algorithms for the minimal 5-point relative pose of perspective cameras, for example [9, 10, 11, 12]. We note that other solutions (that do not explicitly use these properties) were derived, for example [13].

However, and mainly to get wide field of views, new imaging devices have been developed – for example multiple perspective camera systems, catadioptric cameras or cameras with complex optical systems. In most of these cases, camera models are non-central. As a result, all of these methods for relative pose can not be used and new algorithms have to be developed.

To deal with general cases (central and non-central camera models) Pless [14] proposed the concept of the generalized epipolar constraint. He considered that a camera can be represented by the general camera model (proposed by Grossberg and Nayar at [15]), which basically assumes that all pixels are mapped into 3D straight lines in the world. Similarly to the case described in the first paragraph, the match of image pixels is mapped into 3D straight lines and the goal is to estimate the rigid transformation that aligns these 3D lines to ensure that they intersect. To represent lines, Pless used *Plücker* coordinates – a line is represented by $\mathbf{l} \doteq (\mathbf{d}, \mathbf{m}) \subset \mathbb{R}^6$ [16] where $\mathbf{d}, \mathbf{m} \in \mathbb{R}^3$ are the direction and moment of the lines respectively. Under this framework, Pless defined the generalized epipolar constraint as:

$$\mathbf{l}_i^{(2)\,T} \underbrace{\begin{pmatrix} s\left(\mathbf{t}\right)\mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0} \end{pmatrix}}_{\mathcal{E}\subset\mathbb{R}^{6\times 6}} \mathbf{l}_i^{(1)} = 0, \qquad (2)$$

where $\mathcal{E}$ is denoted as generalized essential matrix. From (2), one can see that 17 corresponding 3D lines can be used to compute $\mathcal{E}$ linearly. Sturm at [17] studied the properties of the generalized essential matrix when the underlying camera model belongs to central, axial and xslit cameras, which included the minimum number of correspondences between projection rays required for computing essential matrices using linear equations, for each case. Li et al. at [18] show that despite the rank deficiency in generalized essential matrix for different camera models, it is possible to compute the rotation and translation between two views for different configurations and demonstrated real results on multi-camera configurations. Kim and Kanade at [19] decomposed the generalized essential matrix to study the degenerate cases for specific type of ray geometry.

To conclude, we note that several algorithms for the relative pose under the framework of generalized camera models have been developed: Lhuillier [20] proposed a generic structure-from-motion method based on an angular error; Schweighofer and Pinz [21] proposed a globally convergent solution to the structure and motion estimation; and Stewenius et al. [10] proposed a solution for the minimal 6-point relative pose problem.

In the case of central cameras the essential matrix has been extensively used to estimate relative pose. The generalized epipolar constraint has been less frequently employed to estimate the relative pose. One of the reasons may be linked to the fact the generalized essential matrix has not been analyzed with the same level of detail as the essential matrix for central cameras. One of the goals of this paper is to derive some properties of the generalized essential matrix allowing a deeper understanding of its structure. In particular we derive some properties of the singular value decomposition of $\mathcal{E}$ (which can be compared to the result of Proposition 1 in the case of $\mathbf{E}$) that should be helpful for applications of the generalized essential matrix in relative pose applications (specially for the minimal case). We start by considering the following proposition:

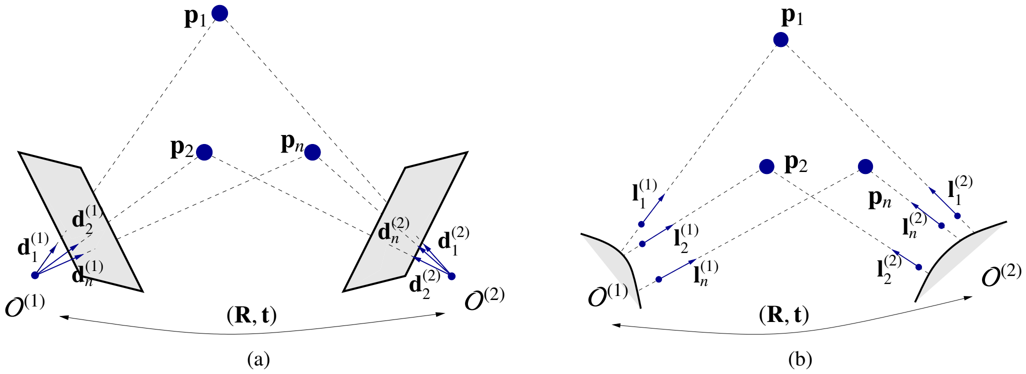

Figure 1: Representation of the relative pose problem for both central (a) and non-central cases (b).

Proposition 3: *Matrix $\mathcal{E}$ is full-rank and its determinant is $det\,(\mathcal{E}) = 1$.*

Proof. Since $\mathcal{E}$ is a block triangular matrix and from [22], $\text{rank}\,(\mathcal{E}) = \text{rank}\,(\mathbf{R}) + \text{rank}\,(\mathbf{R})$ and since $\mathbf{R} \in \mathcal{SO}\,(3)$ implies $\text{rank}\,(\mathbf{R}) = 3$, one can conclude that $\text{rank}\,(\mathcal{E}) = 6$ and that the matrix has *full-rank*. Again, since $\mathcal{E}$ is a block triangular matrix we may write $\det\,(\mathcal{E}) = \det\,(\mathbf{R})\det\,(\mathbf{R})$ and, since $\mathbf{R} \in \mathcal{SO}\,(3)$ implies $\det\,(\mathbf{R}) = 1$, $\det\,(\mathcal{E}) = 1$, proving the proposition. ∎

Let us consider the decomposition denoted as

$$\mathcal{E} \doteq \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \quad \text{where} \;\; \mathbf{U}, \mathbf{V} \in \mathcal{SO}\,(6) \;\; \text{which implies} \;\; \mathbf{U}\mathbf{U}^T = \mathbf{V}\mathbf{V}^T = \mathbf{I} \tag{3}$$

and $\mathbf{\Sigma}$ is a six-dimensional diagonal matrix. For notation, let us consider the representation of the Singular Value Decomposition, such that:

$$\mathcal{E}\mathbf{V} = \mathbf{U}\mathbf{\Sigma} \quad \Rightarrow \quad \mathcal{E}\mathbf{v}_i = \sigma_i \mathbf{u}_i, \quad \forall i = 1, \ldots, 6, \tag{4}$$

where: σ_i, $\mathbf{v}_i$ and $\mathbf{u}_i$ are called the i^{th} singular values, right and left singular vectors respectively. $\mathbf{u}_i$ and $\mathbf{v}_i$ are the columns of $\mathbf{U}$ and $\mathbf{V}$.

The main contributions of the paper are derived in the following section. We propose three theorems, namely: the eigen decomposition of $\mathcal{E}\mathcal{E}^T$ (Theorem 1), the singular value decomposition of $\mathcal{E}$ (Theorem 2), and the sufficient conditions to ensure that a singular value decomposition represents an essential matrix (Theorem 3).

2. Singular Value Decomposition of $\mathcal{E}$

From (3), let us consider the following results:

$$\mathcal{E}\mathcal{E}^T = \mathbf{U}\mathbf{\Sigma}\underbrace{\mathbf{V}^T\mathbf{V}}_{\mathbf{I}}\mathbf{\Sigma}\mathbf{U}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T \quad \text{and} \quad \mathcal{E}^T\mathcal{E} = \mathbf{V}\mathbf{\Sigma}\underbrace{\mathbf{U}^T\mathbf{U}}_{\mathbf{I}}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T, \tag{5}$$

where $\mathbf{\Sigma}^2$ is a diagonal matrix with elements σ_i^2. Taking into account that

$$\mathbf{R} \in \mathcal{SO}\,(3) \Rightarrow \mathbf{R}^T\mathbf{R} = \mathbf{I} \;\; \text{and} \;\; s\,(\mathbf{t})^T = -s\,(\mathbf{t}), \tag{6}$$

we get

$$\mathcal{E}\mathcal{E}^T = \left(\begin{array}{c|c} -s\,(\mathbf{t})\,\mathbf{R}\mathbf{R}^T s\,(\mathbf{t}) + \mathbf{R}\mathbf{R}^T & s\,(\mathbf{t})\,\mathbf{R}\mathbf{R}^T \\ \hline -\mathbf{R}\mathbf{R}^T s\,(\mathbf{t}) & \mathbf{R}\mathbf{R}^T \end{array}\right) = \left(\begin{array}{c|c} -s\,(\mathbf{t})\,s\,(\mathbf{t}) + \mathbf{I} & s\,(\mathbf{t}) \\ \hline -s\,(\mathbf{t}) & \mathbf{I} \end{array}\right) \tag{7}$$

and

$$\mathcal{E}^T\mathcal{E} = \left(\begin{array}{c|c} \mathbf{R}^T\mathbf{R} - \mathbf{R}^T s(\mathbf{t})\, s(\mathbf{t})\,\mathbf{R} & -\mathbf{R}^T s(\mathbf{t})\,\mathbf{R} \\ \hline \mathbf{R}^T s(\mathbf{t})\,\mathbf{R} & \mathbf{R}^T\mathbf{R} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I} - \mathbf{R}^T s(\mathbf{t})\, s(\mathbf{t})\,\mathbf{R} & -\mathbf{R}^T s(\mathbf{t})\,\mathbf{R} \\ \hline \mathbf{R}^T s(\mathbf{t})\,\mathbf{R} & \mathbf{I} \end{array}\right) \tag{8}$$

As a result we define the following proposition:

Proposition 4: *The matrix $\mathcal{E}$, in* (2)*, is such that $\mathcal{E}\mathcal{E}^T$ depends only on the translation vector* $\mathbf{t}$.

Proof. This proposition results from a simple analysis of (7). ∎

2.1. Eigen Decomposition of $\mathcal{E}\mathcal{E}^T$

We will use the result of Proposition 4 to start our derivations. Let us consider the eigen decomposition of the matrix $\mathcal{E}\mathcal{E}^T$ such that

$$\mathcal{E}\mathcal{E}^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i. \tag{9}$$

Proposition 5: *Matrix $\mathcal{E}\mathcal{E}^T$ has three eigenvalues which are equal respectively to σ_+^2,* 1 *and $\left(\sigma_+^2\right)^{-1}$, each of which has algebraic multiplicity of two.*

Proof. To get the respective eigenvalues, we consider the characteristic polynomial of $\mathcal{E}\mathcal{E}^T$, which corresponds to $\det\left(\mathcal{E}\mathcal{E}^T - \sigma^2\mathbf{I}\right) = 0$. Using the fact that $\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{D})\det\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)$, if $\mathbf{D}$ is invertible, one can write

$$\det\left(\mathcal{E}\mathcal{E}^T - \sigma^2\mathbf{I}\right) = \underbrace{\det(\xi\mathbf{I})}_{\xi^3}\det\left(-s(\mathbf{t})^2 + \xi\mathbf{I} + \xi^{-1}s(\mathbf{t})^2\right) \quad \text{where} \quad \xi = \left(-\sigma^2 + 1\right) \tag{10}$$

which can be rewritten as

$$\det\left(\mathcal{E}\mathcal{E}^T - \sigma^2\mathbf{I}\right) = \xi^3\det\left(\xi^{-1}\left(\xi^2\mathbf{I} - \xi s(\mathbf{t})^2 + s(\mathbf{t})^2\right)\right) = \cancel{\xi^3}\cancel{\xi^{-3}}\det\Big(\underbrace{\left(\xi^2\mathbf{I} - \xi s(\mathbf{t})^2 + s(\mathbf{t})^2\right)}_{\mathbf{M}\in\mathbb{R}^{3\times 3}}\Big) \tag{11}$$

As a result, since we want $\det\left(\mathcal{E}\mathcal{E}^T - \sigma^2\mathbf{I}\right) = 0$, the eigenvalues can be easily found by computing $\det(\mathbf{M}) = 0$ which gives

$$\det\left(\mathcal{E}\mathcal{E}^T - \sigma^2\mathbf{I}\right) = 0 \implies \xi^2\left(\xi - \underbrace{\left(-\frac{\mathbf{t}^T\mathbf{t}}{2} - \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}\right)}_{k_+}\right)^2\left(\xi - \underbrace{\left(-\frac{\mathbf{t}^T\mathbf{t}}{2} + \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}\right)}_{k_-}\right)^2 = 0$$

$$\equiv \left(\sigma^2 - 1\right)^2\left(\sigma^2 - (1 - k_+)\right)^2\left(\sigma^2 - (1 - k_-)\right)^2 = \left(\sigma^2 - 1\right)^2\left(\sigma^2 - \sigma_+^2\right)^2\left(\sigma^2 - \sigma_-^2\right)^2 = 0 \tag{12}$$

where

$$\sigma_+^2 = 1 + \frac{\mathbf{t}^T\mathbf{t}}{2} + \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2} \text{ and } \sigma_-^2 = 1 + \frac{\mathbf{t}^T\mathbf{t}}{2} - \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}. \tag{13}$$

From (12), one can conclude the set $\left\{1, 1, \sigma_+^2, \sigma_+^2, \sigma_-^2, \sigma_-^2\right\}$ defines the eigenvalues of $\mathcal{E}\mathcal{E}^T$. Moreover, since each of the eigenvalues is repeated twice, we can write σ_+^2, 1 and σ_-^2 are the respective eigenvalues, each of them has an algebraic multiplicity of two.

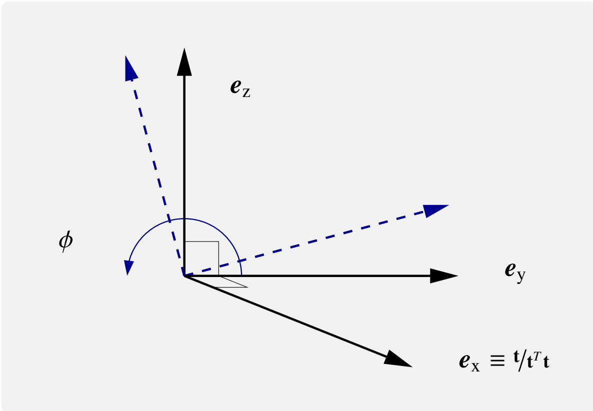

Figure 2: Graphical representation of the basis $\boldsymbol{e}_\mathrm{y}$ and $\boldsymbol{e}_\mathrm{z}$ as a function of the translation vector $\mathbf{t}$. As we can see from the figure, there is a degree of freedom (angle ϕ) associated to the choice of both $\boldsymbol{e}_\mathrm{y}$ and $\boldsymbol{e}_\mathrm{z}$.

Let us now consider the following derivation

$$\sigma_+^2\sigma_-^2 = 1 + \frac{\mathbf{t}^T\mathbf{t}}{2} \cancel{- \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}} + \frac{\mathbf{t}^T\mathbf{t}}{2} \cancel{+ \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}} + \frac{\left(\mathbf{t}^T\mathbf{t}\right)^2}{4} \\ \cancel{+ \frac{\mathbf{t}^T\mathbf{t}}{2}\frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}} \cancel{- \frac{\mathbf{t}^T\mathbf{t}}{2}\frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}} + - \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}\frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}, \quad (14)$$

and, as a result,

$$\sigma_+^2\sigma_-^2 = 1 + \cancel{\frac{\mathbf{t}^T\mathbf{t}}{2}} + \cancel{\frac{\mathbf{t}^T\mathbf{t}}{2}} + \cancel{\frac{\left(\mathbf{t}^T\mathbf{t}\right)^2}{4}} - \cancel{\frac{\left(\mathbf{t}^T\mathbf{t}\right)^2}{4}} + \cancel{\frac{4\mathbf{t}^T\mathbf{t}}{4}} \Rightarrow \sigma_+^2\sigma_-^2 = 1, \quad \text{for any } \mathbf{t}^T\mathbf{t}. \quad (15)$$

With this result, we prove that $\sigma_-^2 = \left(\sigma_+^2\right)^{-1}$, which means that the eigenvalues of $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^T$ are σ_+^2, 1 and $\left(\sigma_+^2\right)^{-1}$, concluding the proof of the proposition. ∎

Let us now consider the computation of the eigenvectors of $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^T$. For that purpose, let us consider the basis vectors $\boldsymbol{e}_\mathrm{y}$ and $\boldsymbol{e}_\mathrm{z}$ as shown in Fig. 2. The derivation of each of the eigenvectors and of the complete eigendecomposition of $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^T$ is shown in the following proposition and theorem:

Proposition 6: *For each eigenvalue defined in Proposition 5, we have the following associated eigenvectors:*

- σ_+^2 *associated with* $\mathbf{u}_1 = \left(\boldsymbol{e}_\mathrm{z}, \zeta_+\boldsymbol{e}_\mathrm{y}\right)$ *and* $\mathbf{u}_2 = \left(-\boldsymbol{e}_\mathrm{y}, \zeta_+\boldsymbol{e}_\mathrm{z}\right)$*;*
- 1 *associated with* $\mathbf{u}_3 = (\mathbf{0}, \mathbf{t})$ *and* $\mathbf{u}_4 = (\mathbf{t}, \mathbf{0})$*; and*
- σ_+^2 *associated with* $\mathbf{u}_5 = \left(\boldsymbol{e}_\mathrm{z}, \zeta_-\boldsymbol{e}_\mathrm{y}\right)$*, and* $\mathbf{u}_6 = \left(-\boldsymbol{e}_\mathrm{y}, \zeta_-\boldsymbol{e}_\mathrm{z}\right)$*,*

for some scalars ζ_+ *and* ζ_- *and basis* $\boldsymbol{e}_\mathrm{y}$ *and* $\boldsymbol{e}_\mathrm{y}$ *as shown in Fig. 2.*

Proof. Let us consider the basis vectors $\boldsymbol{e}_\mathrm{y}$ and $\boldsymbol{e}_\mathrm{z}$ as shown in Fig. 2. From their definition, one can write

$$s(\mathbf{t})\,\boldsymbol{e}_\mathrm{y} = \sqrt{\mathbf{t}^T\mathbf{t}}\boldsymbol{e}_\mathrm{z}, \quad s(\mathbf{t})\,\boldsymbol{e}_\mathrm{z} = -\sqrt{\mathbf{t}^T\mathbf{t}}\boldsymbol{e}_\mathrm{y}, \quad s(\mathbf{t})^2\,\boldsymbol{e}_\mathrm{y} = -\mathbf{t}^T\mathbf{t}\boldsymbol{e}_\mathrm{y}, \quad \text{and} \quad s(\mathbf{t})^2\,\boldsymbol{e}_\mathrm{z} = -\mathbf{t}^T\mathbf{t}\boldsymbol{e}_\mathrm{z}. \quad (16)$$

Let us define

$$\zeta_+ = \frac{\sigma_+^2 - 1 - \mathbf{t}^T\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}} = -\frac{\sqrt{\mathbf{t}^T\mathbf{t}}}{2} + \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2\sqrt{\mathbf{t}^T\mathbf{t}}} \quad \text{and} \quad \zeta_- = \frac{\sigma_-^2 - 1 - \mathbf{t}^T\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}} = -\frac{\sqrt{\mathbf{t}^T\mathbf{t}}}{2} - \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2\sqrt{\mathbf{t}^T\mathbf{t}}}. \tag{17}$$

To prove the proposition, we have to ensure that $\mathcal{E}\mathcal{E}^T\mathbf{u}_1 = \sigma_+^2\mathbf{u}_1$, $\mathcal{E}\mathcal{E}^T\mathbf{u}_2 = \sigma_+^2\mathbf{u}_2$, $\mathcal{E}\mathcal{E}^T\mathbf{u}_3 = \mathbf{u}_3$, $\mathcal{E}\mathcal{E}^T\mathbf{u}_4 = \mathbf{u}_4$, $\mathcal{E}\mathcal{E}^T\mathbf{u}_5 = \sigma_-^2\mathbf{u}_5$ and $\mathcal{E}\mathcal{E}^T\mathbf{u}_6 = \sigma_-^2\mathbf{u}_6$. From (16) and (17), one can derive

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} \boldsymbol{e}_z \\ \hdashline \zeta_+\boldsymbol{e}_y \end{array}\right) = \left(\begin{array}{c} \left(\mathbf{t}^T\mathbf{t} + \zeta_+\sqrt{\mathbf{t}^T\mathbf{t}} + 1\right)\boldsymbol{e}_z \\ \hdashline \left(\sqrt{\mathbf{t}^T\mathbf{t}} + \zeta_+\right)\boldsymbol{e}_y \end{array}\right) = \sigma_+^2\left(\begin{array}{c} \boldsymbol{e}_z \\ \hdashline \zeta_+\boldsymbol{e}_y \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_1 = \sigma_+^2\mathbf{u}_1, \tag{18}$$

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} -\boldsymbol{e}_y \\ \hdashline \zeta_+\boldsymbol{e}_z \end{array}\right) = \sigma_+^2\left(\begin{array}{c} -\boldsymbol{e}_y \\ \hdashline \zeta_+\boldsymbol{e}_z \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_2 = \sigma_+^2\mathbf{u}_2. \tag{19}$$

More easily, one can derive

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{t} \end{array}\right) = \left(\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{t} \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_3 = \mathbf{u}_3 \tag{20}$$

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} \mathbf{t} \\ \hdashline \mathbf{0} \end{array}\right) = \left(\begin{array}{c} \mathbf{t} \\ \hdashline \mathbf{0} \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_4 = \mathbf{u}_4 \tag{21}$$

To conclude,

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} \boldsymbol{e}_z \\ \hdashline \zeta_-\boldsymbol{e}_y \end{array}\right) = \left(\begin{array}{c} \left(\mathbf{t}^T\mathbf{t} + \zeta_-\sqrt{\mathbf{t}^T\mathbf{t}} + 1\right)\boldsymbol{e}_z \\ \hdashline \left(\sqrt{\mathbf{t}^T\mathbf{t}} + \zeta_-\right)\boldsymbol{e}_y \end{array}\right) = \sigma_-^2\left(\begin{array}{c} \boldsymbol{e}_z \\ \hdashline \zeta_-\boldsymbol{e}_y \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_5 = \sigma_-^2\mathbf{u}_5 \tag{22}$$

$$\left(\begin{array}{c:c} -s(\mathbf{t})^2 + \mathbf{I} & s(\mathbf{t}) \\ \hdashline -s(\mathbf{t}) & \mathbf{I} \end{array}\right)\left(\begin{array}{c} -\boldsymbol{e}_y \\ \hdashline \zeta_-\boldsymbol{e}_z \end{array}\right) = \sigma_-^2\left(\begin{array}{c} -\boldsymbol{e}_y \\ \hdashline \zeta_-\boldsymbol{e}_z \end{array}\right) \implies \mathcal{E}\mathcal{E}^T\mathbf{u}_6 = \sigma_-^2\mathbf{u}_6 \tag{23}$$

Since the eigenvectors $\mathbf{u}_i$ form an orthogonal basis in $\mathbf{R}^6$, we have to ensure that $\mathbf{u}_i^T\mathbf{u}_j = 0$, for all $i \neq j$. Since $\mathbf{t}^T\boldsymbol{e}_y = 0$, $\mathbf{t}^T\boldsymbol{e}_z = 0$, $\boldsymbol{e}_y^T\boldsymbol{e}_z = 0$, one can easily verify that $\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_1^T\mathbf{u}_4 = \mathbf{u}_1^T\mathbf{u}_6 = \mathbf{u}_2^T\mathbf{u}_3 = \mathbf{u}_2^T\mathbf{u}_4 = \mathbf{u}_2^T\mathbf{u}_5 = \mathbf{u}_3^T\mathbf{u}_4 = \mathbf{u}_3^T\mathbf{u}_5 = \mathbf{u}_3^T\mathbf{u}_6 = \mathbf{u}_4^T\mathbf{u}_5 = \mathbf{u}_4^T\mathbf{u}_6 = \mathbf{u}_5^T\mathbf{u}_6 = 0$. As a result, to conclude the *proof*, we only have to ensure that $\mathbf{u}_1^T\mathbf{u}_5 = 0$ and $\mathbf{u}_2^T\mathbf{u}_6 = 0$. Let us consider the first case. From (17), one can derive

$$\begin{aligned} \mathbf{u}_1^T\mathbf{u}_5 &= \begin{pmatrix} \boldsymbol{e}_z & \zeta_+\boldsymbol{e}_y \end{pmatrix}\begin{pmatrix} \boldsymbol{e}_z \\ \zeta_-\boldsymbol{e}_y \end{pmatrix} = 1 + \zeta_+\zeta_- = \\ &= 1 + \frac{\mathbf{t}^T\mathbf{t}}{4} + \cancel{\frac{\sqrt{\mathbf{t}^T\mathbf{t}}}{2}\frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2\sqrt{\mathbf{t}^T\mathbf{t}}}} - \cancel{\frac{\sqrt{\mathbf{t}^T\mathbf{t}}}{2}\frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2\sqrt{\mathbf{t}^T\mathbf{t}}}} - \frac{\left(\mathbf{t}^T\mathbf{t}\right)^2 + 4\mathbf{t}^T\mathbf{t}}{4\mathbf{t}^T\mathbf{t}} = \cancel{1} + \cancel{\frac{\mathbf{t}^T\mathbf{t}}{4}} - \cancel{\frac{\mathbf{t}^T\mathbf{t}}{4}} - \cancel{1} \implies \mathbf{u}_1^T\mathbf{u}_5 = 0. \end{aligned} \tag{24}$$

and

$$\mathbf{u}_2^T\mathbf{u}_6 = \begin{pmatrix} -\boldsymbol{e}_y & \zeta_+\boldsymbol{e}_z \end{pmatrix}\begin{pmatrix} -\boldsymbol{e}_y \\ \zeta_-\boldsymbol{e}_z \end{pmatrix} = 1 + \zeta_+\zeta_- \implies \mathbf{u}_2^T\mathbf{u}_6 = 0, \tag{25}$$

concluding the proof of the proposition. ■

Now, using the previous propositions, we can define the complete eigen decomposition of $\mathcal{E}\mathcal{E}^T$.

Theorem 1: *The eigen decomposition of the matrix $\mathcal{E}\mathcal{E}^T$ is given by $\mathcal{E}\mathcal{E}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T$, where $\mathbf{U}$ and $\mathbf{\Sigma}^2$ are* $\mathbf{U} = \left(\frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T\mathbf{u}_1}}, \frac{\mathbf{u}_2}{\sqrt{\mathbf{u}_2^T\mathbf{u}_2}}, \ldots, \frac{\mathbf{u}_6}{\sqrt{\mathbf{u}_6^T\mathbf{u}_6}},\right)$,

$$\begin{array}{l:l:l} \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T\mathbf{u}_1}} = \left(\frac{1}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_z, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_y\right) & \frac{\mathbf{u}_2}{\sqrt{\mathbf{u}_2^T\mathbf{u}_2}} = \left(\frac{-1}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_y, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_z\right) & \frac{\mathbf{u}_3}{\sqrt{\mathbf{u}_3^T\mathbf{u}_3}} = \left(\mathbf{0}, \frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}\right) \\ \frac{\mathbf{u}_4}{\sqrt{\mathbf{u}_4^T\mathbf{u}_4}} = \left(\frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}, \mathbf{0}\right) & \frac{\mathbf{u}_5}{\sqrt{\mathbf{u}_5^T\mathbf{u}_5}} = \left(\frac{1}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_z, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_y\right) & \frac{\mathbf{u}_6}{\sqrt{\mathbf{u}_6^T\mathbf{u}_6}} = \left(\frac{-1}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_y, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_z\right) \end{array} \tag{26}$$

and

$$\mathbf{\Sigma}^2 = \begin{pmatrix} \sigma_+^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_+^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\sigma_+^2\right)^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\sigma_+^2\right)^{-1} \end{pmatrix}. \tag{27}$$

To conclude, $\sigma_+^2 > 1 > \left(\sigma_+^2\right)^{-1}$.

Proof. The proof of this theorem results from Propositions 5 and 6 by stacking the eigenvalues σ_i^2 and eigenvectors $\mathbf{u}_i$. The eigenvector may be defined up to a scale factor. However, it is usual to consider that $\mathbf{U}\mathbf{U}^T = \mathbf{I}$. In addition, it is also common to consider the eigenvalues in decreasing order. Thus, for the proof of this theorem, we have to verify the following conditions:

- The columns of $\mathbf{U}$ are normalized (unit vectors); and
- The eigenvalues are organized in decreasing order $\sigma_+^2 > 1 > \left(\sigma_+^2\right)^{-1}$.

To ensure that the eigenvectors have unit norm, instead of considering $\mathbf{u}_i$, we have used $\frac{\mathbf{u}_i}{\sqrt{\mathbf{u}_i^T \mathbf{u}_i}}$. Deriving the expressions for all i, we get the result shown in (26).

To conclude the proof, we have to ensure that $\sigma_+^2 > 1 > \left(\sigma_+^2\right)^{-1}$, which is the same as to prove that $\sigma_+^2 > \left(\sigma_+^2\right)^{-1}$. Let us rewrite σ_+^2 and σ_-^2 (derived in (13)) as

$$\sigma_+^2 = \mu_1 + \mu_2 \quad \text{and} \quad \sigma_-^2 = \mu_1 - \mu_2, \tag{28}$$

where

$$\mu_1 = 1 + \frac{\mathbf{t}^T\mathbf{t}}{2} \quad \text{and} \quad \mu_2 = \frac{\sqrt{(\mathbf{t}^T\mathbf{t})^2 + 4\mathbf{t}^T\mathbf{t}}}{2}. \tag{29}$$

Since $\mathbf{t}^T\mathbf{t} > 0$, one has $\mu_1 > 0$ and $\mu_2 > 0$ and, from (28),

$$\sigma_+^2 > \sigma_-^2 \quad \text{and, since} \quad \sigma_-^2 = \left(\sigma_+^2\right)^{-1}, \quad \sigma_+^2 > \left(\sigma_+^2\right)^{-1}, \tag{30}$$

concluding the proof of the theorem. ■

From the previous derivations, one can define the following result:

Proposition 7: *For any* $\sigma_+^2 \geq 1$ *and* $\Gamma \in \mathcal{SO}(3)$*, the diagonal matrix* $\mathbf{\Sigma}$ *and matrix* $\mathbf{U}$*, defined as*

$$\mathbf{\Sigma}^2 = \begin{pmatrix} \sigma_+^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_+^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_+^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_+^{-2} \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} \mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sigma_+^{(1)} & 0 & 0 & 0 & -\sigma_+^{(2)} \\ \sigma_+^{(1)} & 0 & 0 & 0 & \sigma_+^{(2)} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sigma_+^{(2)} & 0 & 0 & 0 & -\sigma_+^{(1)} & 0 \\ 0 & \sigma_+^{(2)} & 0 & 0 & 0 & -\sigma_+^{(1)} \end{pmatrix}}_{\mathbf{K} \in \mathbb{R}^{6\times 6}}, \tag{31}$$

define the eigendecomposition of $\mathcal{E}\mathcal{E}^T$ *for some generalized essential matrix* $\mathcal{E}$*, where* $\sigma_+^{(1)} = \sqrt{\frac{\sigma_+^2}{1+\sigma_+^2}}$ *and* $\sigma_+^{(2)} = \sqrt{\frac{1}{1+\sigma_+^2}}$*. To conclude, one can define* $\mathbf{t}^T\mathbf{t} = \frac{(\sigma_+^2-1)^2}{\sigma_+^2}$ *and, as a result,* $\mathbf{t} = \frac{\sigma_+^2-1}{\sigma_+}\boldsymbol{\gamma}_1$*, where* $\boldsymbol{\gamma}_1$ *is the first column of* $\mathbf{\Gamma}$.

Proof. From the previous Theorem, one can see that $\mathbf{\Sigma}^2$ depends only on one parameter. Let us consider random values for σ_+^2, from (13) one can see that

$$\mathbf{t}^T\mathbf{t} = \frac{\left(\sigma_+^2 - 1\right)^2}{\sigma_+^2}. \tag{32}$$

Note that, also from (13), σ_+^2 must be bigger than one. Let us now consider ζ_+. Replacing $\mathbf{t}^T\mathbf{t}$ in (17) using (32), one can derive $\zeta_+ = \sigma_+^{-1}$ and since $\zeta_- = -\zeta_+^{-1}$ (see (24)), $\zeta_- = \sigma_+$. To conclude, the basis $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ can be defined by a rotation matrix $\mathbf{\Gamma} \in \mathcal{SO}(3)$. Using these results on (26), we can derive $\mathbf{U}$ as is shown in (31), which means that the eigen decomposition of $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^T$ is as shown in (31), for some $\mathbf{\Gamma} \in \mathcal{SO}(3)$ and $\sigma_+^2 \geq 1$.

Since $\boldsymbol{e}_x = \frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}$ is equal to the first column of $\mathbf{\Gamma}$ and from (32), one can see that

$$\mathbf{t} = \sqrt{\frac{\left(\sigma_+^2 - 1\right)^2}{\sigma_+^2}}\boldsymbol{\gamma}_1 = \frac{\sigma_+^2 - 1}{\sigma_+}\boldsymbol{\gamma}_1, \tag{33}$$

concluding the *proof* of the proposition. ■

In the next section, we use the Eigen Decomposition of $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^T$ (Theorem 1), to derive the Singular Value Decomposition of $\boldsymbol{\mathcal{E}}$.

2.2. Singular Value Decomposition of $\boldsymbol{\mathcal{E}}$

From (3) and the derivation of (5), one can see that the singular values σ_i and left-singular vectors $\mathbf{u}_i$ were derived in the previous section: matrix $\mathbf{U}$ is as shown in Theorem 1 and

$$\Sigma = \begin{pmatrix} \sigma_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_+^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_+^{-1} \end{pmatrix}. \tag{34}$$

From this result and for the complete determination of the elements of the Singular Value Decomposition, it is only necessary to determine the elements of matrix $\mathbf{V}$. Thus, let us consider the following theorem:

Theorem 2: *The Singular Value Decomposition of $\boldsymbol{\mathcal{E}}$ is given by $\boldsymbol{\mathcal{E}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{\Sigma}$ are shown in Theorem 1 and* (34)*, respectively, and $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6)$, are such that*

$$\sigma_+\mathbf{v}_1^T = \boldsymbol{\mathcal{E}}^T\left(\frac{1}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_z, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_y\right) = \left(-\frac{1}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T s(\mathbf{t})\,\boldsymbol{e}_z + \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T\boldsymbol{e}_y, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T\boldsymbol{e}_z\right) \tag{35}$$

$$\sigma_+\mathbf{v}_2^T = \boldsymbol{\mathcal{E}}^T\left(\frac{-1}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_y, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\boldsymbol{e}_z\right) = \left(\frac{1}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T s(\mathbf{t})\,\boldsymbol{e}_y + \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T\boldsymbol{e}_z, \frac{\zeta_+}{\sqrt{1+\zeta_+^2}}\mathbf{R}^T\boldsymbol{e}_z\right) \tag{36}$$

$$\mathbf{v}_3^T = \boldsymbol{\mathcal{E}}^T\left(\mathbf{0}, \frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}\right) = \left(\mathbf{0}, \frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}\mathbf{R}\right) \tag{37}$$

$$\mathbf{v}_4^T = \boldsymbol{\mathcal{E}}^T\left(\frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}, \mathbf{0}\right) = \left(\frac{\mathbf{t}}{\sqrt{\mathbf{t}^T\mathbf{t}}}\mathbf{R}, \mathbf{0}\right) \tag{38}$$

$$\sigma_+^{-1}\mathbf{v}_5^T = \boldsymbol{\mathcal{E}}^T\left(\frac{1}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_z, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_y\right) = \left(-\frac{1}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T s(\mathbf{t})\,\boldsymbol{e}_z + \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T\boldsymbol{e}_y, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T\boldsymbol{e}_z\right) \tag{39}$$

$$\sigma_+^{-1}\mathbf{v}_6^T = \boldsymbol{\mathcal{E}}^T\left(\frac{-1}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_y, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\boldsymbol{e}_z\right) = \left(\frac{1}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T s(\mathbf{t})\,\boldsymbol{e}_y + \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T\boldsymbol{e}_z, \frac{\zeta_-}{\sqrt{1+\zeta_-^2}}\mathbf{R}^T\boldsymbol{e}_z\right). \tag{40}$$

Proof. As we already mentioned, the only remaining unknown is matrix $\mathbf{V}$. Let us consider the following derivation,

$$\mathbf{U}^T\mathcal{E} = \underbrace{\mathbf{U}^T\mathbf{U}}_{\mathbf{I}}\,\mathbf{\Sigma}\mathbf{V}^T = \mathbf{\Sigma}\mathbf{V}^T, \quad \text{which implies } \; \mathcal{E}^T\mathbf{u}_i = \sigma_i\mathbf{v}_i, \; \forall i = 1,\dots,6. \tag{41}$$

The proof of the theorem is then given by a simple verification of the above condition for each vector $\mathbf{v}_i$. ∎

Let us consider the sufficient conditions for singular value decomposition of $\mathcal{E}$. Let us start by simplifying the results of Theorem 2, using the results of Proposition 7. Using these derivations, the following result can be obtained

$$\mathbf{V}^T = \mathbf{\Sigma}^{-1}\mathbf{U}^T\mathcal{E} = \mathbf{\Sigma}^{-1}\mathbf{K}^T\begin{pmatrix} \mathbf{\Gamma}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}^T \end{pmatrix}\begin{pmatrix} s(\mathbf{t})\mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0} \end{pmatrix} = \mathbf{\Sigma}^{-1}\mathbf{K}^T\begin{pmatrix} \mathbf{\Gamma}^T s(\mathbf{t})\mathbf{R} & \mathbf{\Gamma}^T\mathbf{R} \\ \mathbf{\Gamma}^T\mathbf{R} & \mathbf{0} \end{pmatrix}$$

$$\implies \mathbf{V}^T = \underbrace{\mathbf{\Sigma}^{-1}\mathbf{K}^T\begin{pmatrix} \widetilde{\mathbf{\Gamma}}^T & \mathbf{\Gamma}^T \\ \mathbf{\Gamma}^T & \mathbf{0} \end{pmatrix}}_{\mathbf{A}\in\mathbb{R}^{6\times 6}}\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}, \; \text{where } \widetilde{\mathbf{\Gamma}}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_+^2-1}{\sigma_+} \\ 0 & -\frac{\sigma_+^2-1}{\sigma_+} & 0 \end{pmatrix}\mathbf{\Gamma}^T \tag{42}$$

and $\mathbf{K} \in \mathbb{R}^{6\times 6}$ is defined in (31). From (42), (34) and (31), one can see that $\mathbf{A}$ only depends on the parameters σ_+ and $\mathbf{\Gamma}$. Now we define the following theorem.

Theorem 3: *The parameters $\mathbf{\Sigma}$, $\mathbf{U}$ and $\mathbf{V}$ define a singular value decomposition of a generalized essential matrix $\mathcal{E}$ if, for some $\sigma_+ \geq 1$, $\mathbf{\Gamma} \in \mathcal{SO}(3)$ and $\mathbf{\Delta} \in \mathcal{SO}(3)$, $\mathbf{\Sigma}$ is as shown in* (34)*; $\mathbf{U}$ is as shown in* (31)*; and $\mathbf{V}$ has the form*

$$\mathbf{V}^T = \mathbf{A}\begin{pmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta} \end{pmatrix} \tag{43}$$

($\mathbf{A}$ is derived in (42)*). Moreover, one can see that, for any ϕ, $\widehat{\mathbf{\Gamma}}$ defined as*

$$\widehat{\mathbf{\Gamma}} = \mathbf{\Gamma}\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{pmatrix}, \tag{44}$$

gives the singular value decomposition of the same $\mathcal{E}$ as for $\mathbf{\Gamma}$. Hence, the seven degrees of freedom of the constituents of the singular value decomposition, account, as must be, only for the six degrees of freedom of generalized essential matrices and thus, of their singular value decomposition. To conclude, the translation and rotation parameters are $\mathbf{t} = \frac{\sigma_+^2-1}{\sigma_+}\boldsymbol{\gamma}_1$ and $\mathbf{R} = \mathbf{\Delta}$, respectively.

Proof. The first part of this *proof* results from Proposition 7. From this proposition, one can see that $\mathbf{\Sigma}$ and $\mathbf{U}$ must be as shown in (31) for some $\sigma_+^2 \geq 1$ and $\mathbf{\Gamma} \in \mathcal{SO}(3)$. From the same proposition, we concluded that $\mathbf{t} = \frac{\sigma_+^2-1}{\sigma_+}\boldsymbol{\gamma}_1$, where $\boldsymbol{\gamma}_1$ is the first column of $\mathbf{\Gamma}$. From the definition of $\mathbf{\Sigma}$ and $\mathbf{U}$ and from (42), one can see that $\mathbf{V}$ must be as shown in (43), for some $\mathbf{\Delta} \in \mathcal{SO}(3)$. Using these derivations, from the same equation one can see that the aimed rotation is $\mathbf{R} = \mathbf{\Delta}$.

A simple analysis of these results could indicate that we should have seven degrees of freedom for the definition of the singular value decomposition of $\mathcal{E}$: one for σ_+^2; three for $\mathbf{\Gamma}$; and three for $\mathbf{\Delta}$. Nevertheless from $\mathbf{\Gamma}$, one can see that the second and third columns can be defined up to a rotation parameter–see Fig. 2. Formally, the result of $\mathcal{E} = \mathbf{U}\mathcal{E}\mathbf{V}^T$ using $\widehat{\mathbf{\Gamma}}$, as shown in (44), instead of $\mathbf{\Gamma}$, yields the same $\mathcal{E}$, concluding the proof of this theorem. ∎

3. Conclusions

In this paper, we prove several properties of the generalized essential matrix $\mathcal{E}$. We derive analytically solutions for the Eigen Decomposition of $\mathcal{E}\mathcal{E}^T$ and for the Singular Value Decomposition of $\mathcal{E}$. We prove that the former only depends on the translation parameters and that it can be defined up to a degree of freedom. We prove that both decompositions have only three distinct eigenvalues and singular values and that they depend only on the translation parameters. In addition, we also study the sufficient conditions for both the eigen decomposition of $\mathcal{E}\mathcal{E}^T$ and the singular value decomposition of $\mathcal{E}$. We showed that the latter has six degrees of freedom. All of the results derived in the paper were numerically tested using Matlab and the results will be available on the author's web page.

The goal of this paper is not to estimate $\mathcal{E}$ robustly and accurately. Instead, the goal was to derive properties enabling a better understanding and the development of new robust and accurate algorithms. As a result of these findings, the development of new and more robust algorithms for the estimation of the generalized essential matrix may be possible, by using and enforcing the properties that were proven. Current algorithms for the estimation of the generalized essential matrix do not explicitly enforce the properties proven in these paper. Therefore these results can be used to develop methods that ensure that an estimate for the generalized essential matrix verifies its properties.

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